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Poisson Geometry in Mathematics & Physics
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School on Poisson Geometry
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Conference on Poisson Geometry
July 26th to 30th

POISSON 2010

Yoneda embedding: $\mathcal{A}$, A_∞-cat, $\mathcal{Q} = \text{mod}\,\mathcal{A}$

$\ell : \mathcal{A} \longrightarrow \mathcal{Q} = \text{mod}\,\mathcal{A}$

Y object $\longrightarrow \mathcal{Y} : X \longmapsto \hom_{\mathcal{A}}(X, Y) = \mathcal{Y}(X)$

$\mu^d_{\mathcal{Y}} = \mu^d_{\mathcal{A}} : \mathcal{Y}(X^{d-1}) \otimes \hom_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1)$

$\longrightarrow \mathcal{Y}(X_0)[2-d]$

on morphisms, $\ell^1, \ldots, \ell^d, \ldots$

$\ell^d : \hom_{\mathcal{A}}(Y_{d-1}, Y_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(Y_0, Y_1) \longrightarrow \hom_{\mathcal{Q}}(\mathcal{Y}_0, \mathcal{Y}_d)$

$\ell^1(c)$, for $c \in \hom_{\mathcal{A}}(Y_0, Y_1)$ is the pre-module homom.

$\mathcal{Y}_0(X_{d-1}) \otimes \hom_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \longrightarrow \mathcal{Y}_1(X_0)$

$(b, a_{d-1}, \ldots, a_1) \longmapsto \mu^{d+1}_{\mathcal{A}}(c, b, a_{d-1}, \ldots, a_1)$

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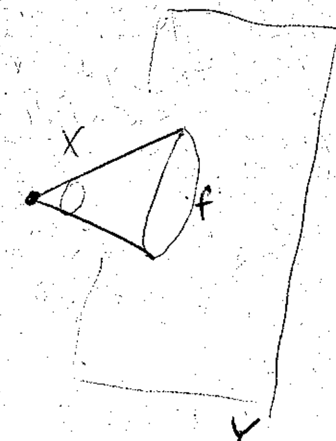
Cones

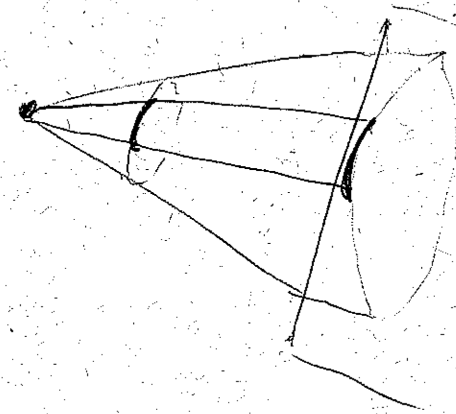
Ex1: Topology: $X \xrightarrow{f} Y \xrightarrow{i} Cone(f) \longrightarrow \Sigma X \longrightarrow$



$$C_i(Cone(f)) \cong C_{i-1}(X) \oplus C_i(Y)$$

$$X \times [0,1] \sqcup Y \Big/ \begin{array}{l}(x,0)\sim(x_0,0)\\ (x,1)\sim f(x)\end{array}$$

$$d = \left(\begin{array}{c|c} \partial_X & \\ \hline f & \partial_Y \end{array}\right)$$



Ex2: Cat. of Dif graded modules ~~out~~ over A;

$M \circlearrowleft d$ $\quad M^i \xrightarrow{d} M^{i+1}$, $d^2=0$

$A \otimes M$, satisfies Leib. rule.

Morfism: $f: M \longrightarrow N$, deg 0,

chain map $\quad d_N f \pm f d_M = 0$

$C = \mathrm{cone}(f) = M[1] \oplus N \;;\quad d_C = \left(\begin{array}{c|c} d_A & 0 \\ \hline f & d_B \end{array}\right)$

$C^i = M^{i+1} \oplus N^i$

$\mathcal{A}$ be A_∞ ~~cat~~; Y_0, Y_1 obj. $c \in \hom^0_{\mathcal{A}}(Y_0, Y_1)$

degree 0 cocycle $(\mu^1_{\mathcal{A}}(c) = 0)$

$\mathcal{C} = \mathrm{Cone}(c)$ is an A_∞ module

$$\mathcal{C}(X) = \hom_{\mathcal{A}}(X, Y_0)[1] \oplus \hom_{\mathcal{A}}(X, Y_1)$$

$\mu^d_{\mathcal{C}}((b_0, b_1), a_{d-1}, \dots, a_1) = (\mu^d_{\mathcal{A}}(b_0, a_{d-1}, \dots, a_1);$

$\cancel{\mu^d_{\mathcal{A}}(b_1, \dots)}$ $\qquad \mu^d_{\mathcal{A}}(b_1, a_{d-1}, \dots, a_1) + \mu^{d+1}_{\mathcal{A}}(c, b_0, a_{d-1}, \dots, a_1))$

$\mu^d_{\mathcal{C}} : \mathcal{C}(X_{d-1}) \otimes \hom_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \dots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \mathcal{C}(X_0)[2-d]$

$$\mu^1_{\mathcal{C}}((b_0, b_1)) = (\mu^1_{\mathcal{A}}(b_0), \mu^1_{\mathcal{A}}(b_1) + \mu^2_{\mathcal{A}}(c, b_0))$$

- $\mathcal{A}$ is c unital $\Longrightarrow$ $\mathcal{C}$ is c-unital module
- $\tilde{c} = c + \mu^1_{\mathcal{A}}(h) \Rightarrow \exists$ ~~[illegible]~~ module hom $\mathrm{Cone}(c) \to \mathrm{Cone}(\tilde{c})$ inducing isom. in $H^0(\text{mu-mod}(\mathcal{A}))$.
- $\mathrm{Cone}(c)$ in A_∞ is any object, quasi representing $\mathrm{Cone}(c)$.

Exact triangles: $\mathcal{Q} = \text{nu-mod}(\mathcal{A})$

$i \in hom^0_{\mathcal{Q}}(Y_1, \mathcal{C})$, $\pi \in hom^1_{\mathcal{Q}}(\mathcal{C}, Y_0)$

$i^1(b_1) = (0, (-1)^{|b_1|} b_1)$; $\pi^1(b_0, b_1) = (-1)^{|b_0|-1} b_0$

$i^d = 0 = \pi^d$

Get in $H(\mathcal{Q})$:

$$\begin{tikzcd} Y_0 \arrow[rr, "{[\ell^1(c)]}"] & & Y_1 \arrow[dl, "{[i]}"] \\ & \mathcal{C} \arrow[ul, "{[\pi]}", "{[1]}"'] & \end{tikzcd} \qquad (*)$$

<u>Def</u> <u>Exact triangle</u> Any diagram in $H(\mathcal{A})$ of the form

$$\begin{tikzcd} Y_0 \arrow[rr, "{[c_1]}"] & & Y_1 \arrow[dl, "{[c_2]}"] \\ & Y_2 \arrow[ul, "{[c_3]}", "{[1]}"'] & \end{tikzcd}$$

isomorphic to $(*)$ via Yoneda.

This means, set $c = c_1$, $[t] : Y_2 \xrightarrow{\text{iso}} \mathcal{C}$ in $H^0(\mathcal{Q})$

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s.t $[\pi]\circ[t] = [\ell^1(c_3)]$ and $[i] = [t]\circ[\ell^1(c_2)]$.

- Independent of c.

<u>Another Caracterization</u>

Consider strictly unital A_∞-Cat $\mathcal{D}$, with objects Z_0, Z_1, Z_2:

$$\hom_{\mathcal{D}}(Z_K, Z_K) = \mathbb{K}\, e_{Z_K}$$

$$\hom_{\mathcal{D}}(Z_1, Z_2) = \mathbb{K}\cdot x_2, \quad |x_2| = 0$$

$$\hom_{\mathcal{D}}(Z_0, Z_1) = \mathbb{K}\cdot x_1, \quad |x_1| = 0$$

$$\hom_{\mathcal{D}}(Z_2, Z_0) = \mathbb{K}\cdot x_3, \quad |x_3| = 1$$

$$\hom_{\mathcal{D}}(Z_2, Z_1) = \hom_{\mathcal{D}}(Z_0, Z_2) = \hom_{\mathcal{D}}(Z_1, Z_0) = 0$$

$$\begin{tikzcd} & Z_2 \arrow[dr, "x_3", "{[1]}"'] & \\ Z_1 \arrow[ur, "x_2"] & & Z_0 \arrow[ll, "x_1"] \end{tikzcd}$$

Only nontrivial composition maps

$$\mu^3_{\mathcal{D}}(x_3, x_2, x_1) = e_{Z_0}, \quad \mu^3_{\mathcal{D}}(x_1, x_3, x_2) = e_{Z_1}, \quad \mu^3_{\mathcal{D}}(x_2, x_1, x_3) = e_{Z_2}$$

<u>Prop</u>: A triangle in $H(\mathcal{A})$ is exact iff $\exists$ A_∞-functor $\mathcal{F}: \mathcal{D} \to \mathcal{A}$ s.t. $\mathcal{F}(Z_K) = Y_K$ and $[\mathcal{F}^1(x_K)] = [c_K]$.

Cor: $F: \mathcal{A} \to \mathcal{B}$ A_∞ func. $\Rightarrow$ $H(F)$ maps exact triangle to exact triangle.

Triangulated A_∞-Cat: $Ob(\mathcal{A}) \neq \emptyset$

and

- $\forall$ morphism $[a]$ in $H^0(\mathcal{A})$ can be completed to an exact triangle.
- $\forall$ obj Y, exist $\tilde{Y}$ s.t $S\tilde{Y} = Y$ in $H^0(\mathcal{A})$

<u>Rmk</u>: Existence of cones $\Rightarrow$ Existence of shift.

<u>Example</u>: A_∞ category $\mathcal{Q} = {}^{nu}\mathrm{mod}\,\mathcal{A}$ are triangulated.

<u>Prop</u> (Just say): $\mathcal{A}, \mathcal{B}$ are A_∞-cat triang. $\Rightarrow$ $H^0(\mathcal{A})$ is triang. as usual cat.

$F : \mathcal{A} \to \mathcal{B}$ A_∞-funct $\Rightarrow$ $H^0(F)$ is exact functor

<u>Properties</u>:

$$\begin{tikzcd} Y_0 \arrow[r] & Y_1 \arrow[dl] \\ Y_2 \arrow[u, "{[1]}"] & \end{tikzcd}$$

$$1) \quad \begin{tikzcd} \mathrm{Hom}_{H(\mathcal{A})}(X, Y_0) \arrow[r] & \mathrm{Hom}_{H(\mathcal{A})}(X, Y_1) \arrow[r] & \mathrm{Hom}_{H(\mathcal{A})}(X, Y_2) \arrow[ll, bend left, "{[1]}"] \end{tikzcd}$$

b) $$\begin{tikzcd} \mathrm{Hom}_{H(A)}(Y_0, X) \arrow[rr, bend right, "{[1]}"'] & \mathrm{Hom}_{H(A)}(Y_1, X) \arrow[l] & \mathrm{Hom}_{H(A)}(Y_2, X) \arrow[l] \end{tikzcd}$$

2) $$\begin{tikzcd} & Y_0 \arrow[rr, "{[b_0]}"] \arrow[dd, "{[c_1]}"] & & \tilde{Y}_0 \arrow[dd, "{[\tilde{c}_1]}"] \\ Y_2 \arrow[ur, "{[c_3]}"] \arrow[rr, dashed, "\exists\,{[b_2]}"] & & \tilde{Y}_2 \arrow[ur, "{[\tilde{c}_3]}"] & \\ & Y_1 \arrow[ul, "{[c_2]}"] \arrow[rr, "{[b_1]}"'] & & \tilde{Y}_1 \arrow[ul, "{[\tilde{c}_2]}"] \end{tikzcd}$$

Moreover,
$[b_0], [b_1]$ isom.
$\Rightarrow [b_2]$ isom.

3) $$\begin{tikzcd} Y_1 \arrow[rr, "{[c_2]}"] & & Y_2 \arrow[dl, "{[c_3]}"] \\ & SY_0 \arrow[ul, "{[1]}", "{-[S(c_1)]}"'] & \end{tikzcd}$$

4) Octahedron prop.

Generators: B triang. A_∞-Cat, $A \subseteq B$ full A_∞ subcat.

The smallest $\tilde{B} \subseteq B$, triang. A_∞ subcat containing A, closed under isom ($X_0 \cong X_1$ in $H(B)$, $X_0 \in \tilde{B} \Rightarrow X_1 \in \tilde{B}$) is said to be generated by A.

Triangulated envelope (B, F) of A: B triang. A_∞-cat

$F: A \to B$ cohom. full and faithfull functor, whose image objects generate B.

<u>Lemma</u>: Triang. Envelope exist. $(\mathcal{B}, F)$; $(\tilde{\mathcal{B}}, \tilde{F})$ such envelopes of $\mathcal{A}$, $\exists$ quasi-equivalence $G: \mathcal{B} \to \tilde{\mathcal{B}}$ s.t $G \circ F \cong \tilde{F}$ in $H^0(\mathrm{fun}(\mathcal{A}, \tilde{\mathcal{B}}))$.

We call $H^0(\mathcal{B}) = D\mathcal{A}$ derived category of $\mathcal{A}$ (independent of $(\mathcal{B}, F)$ up to exact equivalence)

Our favorite choice of envelope: $Tw\,\mathcal{A}$

<u>Rmk</u>: Another choice is full subcat of mod $\mathcal{A}$ generated by image of Yoneda embedding.

<u>Additive Enlargement</u>: $\Sigma\mathcal{A}$, objects $\bigoplus_{i \in I} X^i[n_i]$
finite

Morphisms: $\hom_{\Sigma\mathcal{A}}\left(\bigoplus_{i \in I_0} X_0^i[n_i], \bigoplus_{j \in I_1} X_1^j[m_j]\right) =$

$= \bigoplus_{ij} \hom_{\mathcal{A}}(X_0^i, X_1^j)[m_j - n_i]$

Write morphisms as matrices (a^{ji})

$$\mu^d_{\Sigma A}(a_d, \ldots, a_1)^{kl} = \sum_{i_1, \ldots, i_{d-1}} \pm\, \mu^d_A\left(a_d^{k, i_{d-1}}, a_{d-1}^{i_{d-1}, i_{d-2}}, \ldots, a_1^{i_1, l}\right)$$

<u>Twisted Complex:</u> $\cancel{\left(X = \bigoplus_i X_i[n_i],\ \delta_X = \delta_X^{ij}\right)}$

$\cancel{X \in \mathrm{Ob}\,\Sigma A, \quad \delta_X = \hom^1_{\Sigma A}(X, X)}$

$$\begin{tikzcd} X_1[n_1] \arrow[r, "\delta^{12}"] \arrow[rr, bend right, "\delta^{13}"'] \arrow[rrr, bend right=40, "\delta^{14}"'] & X_2[n_2] \arrow[r, "\delta^{23}"] \arrow[rr, bend right, "\delta^{24}"] & X_3[n_3] \arrow[r] & \cdots \end{tikzcd}$$

$X \in \mathrm{Ob}\,\Sigma A, \quad \delta_X = \hom^1_{\Sigma A}(X, X)$;

$$X = \bigoplus_i X_i[n_i] \quad ; \quad [\delta_X]_{ij} = \begin{bmatrix} 0 & \delta^{12} & \cdots & \delta^{1,n} \\ & \ddots & * & \vdots \\ & 0 & \ddots & \delta^{n-1,n} \\ & & & 0 \end{bmatrix}$$

$$\begin{tikzcd} X_1[n_1] \arrow[r, "\delta^{12}"] \arrow[rr, bend right, "\delta^{13}"] \arrow[rrr, bend right=40, "\delta^{14}"'] & X_2[n_2] \arrow[r, "\delta^{23}"] \arrow[rr, bend left, "\delta^{24}"] & X_3[n_3] \arrow[r] & \cdots \end{tikzcd}$$

Satisfying, Maurer-Cartan eq $\sum_{n=1}^{\infty} \mu^n_{\Sigma A}(\delta_X, \ldots, \delta_X) = 0$;

Twisted complexes in $\mathcal{A}$ for an A_∞-category $Tw\mathcal{A}$.

$$\hom_{Tw\mathcal{A}}(X_0, X_1) = \hom_{\Sigma\mathcal{A}}(X_0, X_1)$$

but compositions are deformed by differentials

$$\mu^d_{Tw\mathcal{A}}(a_d, \dots, a_1) = \sum_{i_0, \dots, i_d} \mu^{d+i_0+\dots+i_d}_{\Sigma\mathcal{A}}(\underbrace{\delta_{X_d}, \dots, \delta_{X_d}}_{i_d}, a_d, \underbrace{\delta_{X_{d-1}}, \dots, \delta_{X_{d-1}}}_{i_{d-1}}, a_{d-1}, \dots, a_1, \underbrace{\delta_{X_0}, \dots, \delta_{X_0}}_{i_0})$$

- It's functorial: $\mathcal{G}: \mathcal{A} \to \mathcal{B}$ induces a functor $Tw\mathcal{G}: Tw\mathcal{A} \longrightarrow Tw\mathcal{B}$
- If $\mathcal{A}$ is c-unital, so is $Tw\mathcal{A}$
- If $\mathcal{G}$ is a quasi-equivalence, then $Tw\mathcal{G}$ also is.

Shift on $Tw\mathcal{A}$: $X = \sum_i X_i[n_i]$; $S^\sigma X = \sum_i X_i[n_i+\sigma]$

Mapping cones: Let $c \in hom^0_{Tw\mathcal{A}}(Y_0, Y_1)$, $\mu^1_{Tw\mathcal{A}}(c) = 0$.

The mapping cone $C = cone(c)$ is the twisted ex.

$$\left(C = Y_0[1] \oplus Y_1 \quad ; \quad \delta_C = \begin{pmatrix} \delta_{Y_0} & 0 \\ c & \delta_{Y_1} \end{pmatrix} \right)$$

Image under Yoneda is ~~isom~~ ident. to $\mathcal{C} = con(c)$, module:

$$hom_{Tw\mathcal{A}}(X, C) = hom_{Tw\mathcal{A}}(X, Y_0[1]) \oplus hom_{Tw\mathcal{A}}(X, Y_1) =$$

$$\cong hom_{Tw\mathcal{A}}(X, Y_0)[1] \oplus hom_{Tw\mathcal{A}}(X, Y_1) = \mathcal{C}(X).$$

Exact triangles in $Tw\mathcal{A}$:

Canonical morphisms $[i] \in Hom^0_{H(Tw\mathcal{A})}(Y_1, C)$; $i = \begin{pmatrix} 0 \\ e_{Y_1} \end{pmatrix}$

$[p] \in Hom^1_{H(Tw\mathcal{A})}(C, Y_0)$; $p = (Se_{Y_0}, 0)$

An triangle in ~~$H(Tw\mathcal{A})$~~ $H^0(Tw\mathcal{A})$ is exact iff $\exists$ isomorphism

$$\begin{tikzcd} Y_0 \arrow[r, "f"] & Y_1 \arrow[dl, "g"] \\ Y_2 \arrow[u, "h"', "{[1]}"] & \end{tikzcd}$$

$[b] : Y_2 \longrightarrow Cone(f)$ in $H^0(Tw\mathcal{A})$ s.t.

$[\mu^2_{Tw\mathcal{A}}(p, b)] = [h]$ and $[i] = [\mu^2_{Tw\mathcal{A}}(b, g)]$

$$\begin{tikzcd} Y_0 \arrow[r, "f"] & Y_1 \arrow[dl, "g"] \\ Y_2 \arrow[u, "h", "{[1]}"'] & \end{tikzcd}$$

get maps

$$\begin{tikzcd} & Y_2 \arrow[dl, "h"'] \arrow[dr, "\hat h"] & \\ Y_0[1] \arrow[rr, "\cancel{f}", "{\begin{bmatrix} \cancel{e_0} & 0 \\ f & \cancel{e_1} \end{bmatrix}}"'] & & Y_1 = \mathrm{Cone}(f) \arrow[dl, "g"] \arrow[dl, bend left, "\hat g"] \\ & Y_2 & \end{tikzcd}$$

$\delta_{Y_0} = \cancel{e_0}$

$\delta_{Y_1} = \cancel{e_1}$

$$\mu^2_{TwA}(\hat h, \hat g) = \mu^3(g, f, h) = e_{Y_2}$$

- TwA is triangulated category
- TwA is generated by it's full subcategory A.
- An A_∞-category A is triangulated $\Leftrightarrow$ $A \hookrightarrow TwA$ is ~~a~~ quasi-equivalence.